Many uses of **linear programming, mixed integer (linear) programming** in this course

<table>
<thead>
<tr>
<th>Linear programming</th>
<th>Mixed integer linear programming</th>
</tr>
</thead>
<tbody>
<tr>
<td>Game theory</td>
<td>Dominated strategies, Minimax strategies, Correlated equilibrium, Nash equilibrium, Optimal mixed strategies to commit to</td>
</tr>
<tr>
<td>Social choice, expressive marketplaces</td>
<td>Winner determination in auctions, exchanges, … with partially acceptable bids, Winner determination in auctions, exchanges, … without partially acceptable bids; Kemeny, Slater, other voting rules; kidney exchange</td>
</tr>
<tr>
<td>Mechanism design</td>
<td>Automatically designing optimal mechanisms that use randomization, Automatically designing optimal mechanisms that do not use randomization</td>
</tr>
</tbody>
</table>

**Properties of Linear Programming (LP) Models**

1) Seek to minimize or maximize
2) Include “constraints” or limitations
3) There must be alternatives available
4) All equations are linear

**Example**

Two products: Chairs and Tables

Decision: How many of each to make this month?

Objective: Maximize profit

**Flair Furniture Co. Data**

<table>
<thead>
<tr>
<th>Tables (per table)</th>
<th>Chairs (per chair)</th>
<th>Hours Available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Profit Contribution</td>
<td>$7</td>
<td>$5</td>
</tr>
<tr>
<td>Carpentry Req</td>
<td>3 hrs</td>
<td>4 hrs</td>
</tr>
<tr>
<td>Painting Req</td>
<td>2 hrs</td>
<td>1 hr</td>
</tr>
</tbody>
</table>

Other Limitations:
- Make no more than 450 chairs
- Make at least 100 tables

**Brief introduction to linear and mixed integer programming**
Decision Variables:

- \( T \) = Num. of tables to make
- \( C \) = Num. of chairs to make

Objective Function: Maximize Profit

Maximize \( 7T + 5C \)

Constraints:

- Have 2400 hours of carpentry time available
  \( 3T + 4C \leq 2400 \) (hours)
- Have 1000 hours of painting time available
  \( 2T + 1C \leq 1000 \) (hours)

More Constraints:

- Make no more than 450 chairs
  \( C \leq 450 \) (num. chairs)
- Make at least 100 tables
  \( T \geq 100 \) (num. tables)

Nonnegativity:

Cannot make a negative number of chairs or tables

- \( T \geq 0 \)
- \( C \geq 0 \)

Model Summary

Maximize \( 7T + 5C \) (profit)

Subject to the constraints:

- \( 3T + 4C \leq 2400 \) (carpentry hrs)
- \( 2T + 1C \leq 1000 \) (painting hrs)
- \( C \leq 450 \) (max # chairs)
- \( T \geq 100 \) (min # tables)
- \( T, C \geq 0 \) (nonnegativity)

Graphical Solution

- Graphing an LP model helps provide insight into LP models and their solutions.

While graphing can only be done in two or three dimensions, the same properties apply to all LP models and solutions.


**LP Characteristics**

- **Feasible Region**: The set of points that satisfies all constraints
- **Corner Point Property**: An optimal solution must lie at one or more corner points
- **Optimal Solution**: The corner point with the best objective function value is optimal

**In higher dimensions**

- In multiple dimensions, the region of interest is of higher dimension.
- **polytope** – a polygon in higher dimension - a geometric object with flat sides
- **simplex** – a generalization of the notion of a triangle or tetrahedron to arbitrary dimension – a convex hull
- **Algorithmic method** – optimal value will always be on a vertex of the polytope – walking along edges to vertices of higher objective function
Linear programs: example

- We make reproductions of two paintings

\[
\begin{align*}
\text{maximize} & \quad 30x + 20y \\
\text{subject to} & \quad 4x + 2y \leq 16 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0, \quad y \geq 0
\end{align*}
\]

- Painting 1 sells for $30, painting 2 sells for $20
- Painting 1 requires 4 units of blue, 1 green, 1 red
- Painting 2 requires 2 blue, 2 green, 1 red
- We have 16 units blue, 8 green, 5 red

Solving the linear program graphically

- Think of dotted lines as various values for \(30x + 20y\)

Optimal solution: \(x = 2.5, \ y = 2.5\) (objective 125)

Modified LP

\[
\begin{align*}
\text{maximize} & \quad 30x + 20y \\
\text{subject to} & \quad 4x + 2y \leq 15 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0, \quad y \geq 0
\end{align*}
\]

Optimal solution: \(x = 2.5, \ y = 2.5\)

Solution value = \(7.5 + 5 = 12.5\)

Half paintings? – interested in integer answers

Integer (linear) program

\[
\begin{align*}
\text{maximize} & \quad 30x + 20y \\
\text{subject to} & \quad 4x + 2y \leq 15 \\
& \quad x + 2y \leq 8 \\
& \quad x + y \leq 5 \\
& \quad x \geq 0, \quad y \geq 0
\end{align*}
\]

Optimal IP solution: \(x = 2, \ y = 3\) (objective 120)

Optimal LP solution: \(x = 2.5, \ y = 2.5\) (objective 125)
Mixed integer (linear) program

maximize \( 30x + 20y \)
subject to
\( 4x + 2y \leq 15 \)
\( x + 2y \leq 8 \)
\( x + y \leq 5 \)
\( x \geq 0 \)
\( y \geq 0, \text{ integer} \)

optimal IP solution: \( x=2, y=3 \) (objective 120)

optimal LP solution: \( x=2.5, y=2.5 \) (objective 125)

optimal MIP solution: \( x=2.75, y=2 \) (objective 122.5)

Solving linear/integer programs

• Linear programs can be solved efficiently
  – Simplex, ellipsoid, interior point methods…

• (Mixed) integer programs are NP-hard to solve
  – Quite easy to model many standard NP-complete problems as integer programs (try it!)
  – Search type algorithms such as branch and bound

• Standard packages for solving these
  – GNU Linear Programming Kit, CPLEX, …

• LP relaxation of (M)IP: remove integrality constraints
  – Gives upper bound on MIP (~admissible heuristic)

Practice modeling hard problems as LP or (M)IP problems

Exercise in modeling: knapsack-type problem
• We arrive in a room full of precious objects
• Can carry only 30kg out of the room
• Can carry only 20 liters out of the room
• Want to maximize our total value
• Unit of object A: 16kg, 3 liters, sells for $11
  – There are 3 units available
• Unit of object B: 4kg, 4 liters, sells for $4
  – There are 4 units available
• Unit of object C: 6kg, 3 liters, sells for $9
  – Only 1 unit available
• What should we take?

Exercise in modeling: cell phones (set cover)
• We want to have a working phone in every continent (besides Antarctica)
• … but we want to have as few phones as possible
• Phone A works in NA, SA, Af
• Phone B works in E, Af, As
• Phone C works in NA, Au, E
• Phone D works in SA, As, E
• Phone E works in Af, As, Au
• Phone F works in NA, E

Exercise in modeling: hot-dog stands
• We have two hot-dog stands to be placed in somewhere along the beach
• We know where the people that like hot dogs are, how far they are willing to walk
• Where do we put our stands to maximize #hot dogs sold? (price is fixed)