Normal-form games

"Chicken"
- Two players drive cars towards each other
- If one player goes straight, that player wins
- If both go straight, they both die

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<tbody>
<tr>
<td>D</td>
<td>0, 0</td>
<td>-1, 1</td>
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<tr>
<td>S</td>
<td>-1, -1</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

not zero-sum

Prisoner's Dilemma
- Pair of criminals has been caught
- District attorney has evidence to convict them of a minor crime (1 year in jail); knows that they committed a major crime together (3 years in jail) but cannot prove it
- Offers them a deal:
  - If both confess to the major crime, they each get a 1 year reduction
  - If only one confesses, that one gets a 3 years reduction

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
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</thead>
<tbody>
<tr>
<td>confess</td>
<td>-2, -2</td>
<td>0, -3</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-3, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

Dominance
- Player i’s strategy $s_i$ strictly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
- $s_i$ weakly dominates $s_i'$ if
  - for any $s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$; and
  - for some $s_{-i}$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$

Rock-paper-scissors
- Row player aka. player 1 (simultaneously) chooses a row
- Column player aka. player 2 (simultaneously) chooses a column

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<thead>
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<tbody>
<tr>
<td>S</td>
<td>0, 0</td>
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</tr>
<tr>
<td>D</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

A row or column is called an action or (pure) strategy

Row player’s utility is always listed first, column player’s second

Zero-sum game: the utilities in each entry sum to 0 (or a constant)

Three-player game would be a 3D table with 3 utilities per entry, etc.

"Chicken"

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<thead>
<tr>
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<tbody>
<tr>
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<tr>
<td>S</td>
<td>-1, -1</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

not zero-sum

"Should I buy an SUV?"
- Purchasing cost: 5, accident cost: 3
- Cost of buying an SUV dominates buying other

<table>
<thead>
<tr>
<th></th>
<th>purchasing cost</th>
<th>accident cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>cost: 5</td>
<td>-10, -10</td>
<td>-7, -11</td>
</tr>
<tr>
<td>cost: 3</td>
<td>-11, -7</td>
<td>-8, -8</td>
</tr>
<tr>
<td>cost: 5</td>
<td>-10, -10</td>
<td>-7, -11</td>
</tr>
</tbody>
</table>
Checking for dominance by mixed strategies

- Linear program (LP) for checking whether strategy $s_i^*$ is strictly dominated by a mixed strategy:
  - select $p_i$ to maximize $\varepsilon$  
  - such that:
    - for any $s_i$, $\sum p_i u(s_i, s_j) \geq u(s_i^*, s_j) + \varepsilon$  
    - $\sum p_i = 1$

- Linear program for checking whether strategy $s_i^*$ is weakly dominated by a mixed strategy:
  - maximize $\sum ([\sum p_i u(s_i, s_j) - u(s_i^*, s_j)]$
  - such that:
    - for any $s_i$, $\sum p_i u(s_i, s_j) \geq u(s_i^*, s_j)$
    - $\sum p_i = 1$

Dominated by a mixed strategy

- If a pure strategy is dominated by a mixed strategy, it may not be immediately obvious. Below, we show only the row player’s payoff. What is dominated?

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Terms

- zero sum $u_1(o) = -u_2(o)$
- constant sum $u_1(o) + u_2(o) = c$
- common payoff (pure coordination or team games) - for the same outcome, agents get the same payoff $u_1(o) = u_2(o)$
- Expected utility of a mixed strategy – sum of probabilities of each outcome times utility of each outcome

Iterated dominance

- removing weakly or very weakly dominated strategies CAN eliminate some equilibria of the original game.
- There is still a computational benefit. No new equilibria are created and at least one of the original equilibria always survives.

Iterated dominance on Seinfeld’s RPS (nothing beats a rock):

<table>
<thead>
<tr>
<th></th>
<th>0,0</th>
<th>1,-1</th>
<th>1,-1</th>
<th>0,0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1,1</td>
<td>0,0</td>
<td>-1,1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

Terms

- positive affine transformation: For positive constant $A$ and offset $b$
  $x \rightarrow Ax + b$

  Ratio of distances along a line are preserved

  If we transform the payoffs with an affine transformation, the solution doesn’t change
Can some outcomes be declared to be better?

- Might be tempting to try to optimize maximum welfare, but any positive affine transformation applied to payoffs, yields another valid utility function. So just maximizing social welfare is bad if agents have different scales.
- Pareto dominance gives us a partial order over strategy profiles. Clearly 4,5 > 3,4 BUT which is better 4,6 or 9,2

Two computational questions for iterated dominance

- 1. Can a given strategy be eliminated using iterated dominance?
- 2. Is there some path of elimination by iterated dominance such that only one strategy per player remains?

- For strict dominance (with or without dominance by mixed strategies), both can be solved in polynomial time due to path-independence:
  - Check if any strategy is dominated, remove it, repeat
- For weak dominance, both questions are NP-hard (even when all utilities are 0 or 1), with or without dominance by mixed strategies [Conitzer, Sandholm 05]

Explain why the problems are NP hard

Lots of complexity issues

- Issues of complexity are often stated
- The Shoham, Leyton-Brown (link on website) talks about many of these, for those of you who are interested.

Can you explain?

- If I am mixing three strategies, you will be mixing three strategies. Why?
- To test a mixed strategy, you see what each of the pure strategies (in the opponents support) do against it. They need to yield the same response. Why don’t you have to check all the mixed strategies?

Iterated dominance: path (in)dependence

Iterated weak dominance is path-dependent: particular sequence of eliminations may determine which solution we get (if any)

Iterated strict dominance is path-independent: elimination process will always terminate at the same point (whether or not dominance by mixed strategies allowed)

Two-player zero-sum games revisited

- Recall: in a zero-sum game, payoffs in each entry sum to zero
  - ... or to a constant: recall that we can subtract a constant from anyone’s utility function without affecting their behavior
- What the one player gains, the other player loses
Two-player general-sum game

Note: a general-sum k-player game can be modeled as a zero-sum (k+1)-player game by adding a dummy player (with no choices) absorbing the remaining utility, so zero-sum games with 3 or more players have to deal with the difficulties of general-sum games; this is why we focus on 2-player zero-sum games here.

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<thead>
<tr>
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<th>2, 1, -3</th>
<th>1, 3, -4</th>
<th>3, 1, -4</th>
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<tbody>
<tr>
<td>2</td>
<td>3, 1, -4</td>
<td>2, 0, -2</td>
<td>1,3, -4</td>
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<tr>
<td>1</td>
<td>3, 1, -4</td>
<td>3, 1, -4</td>
<td>2, 2, -4</td>
</tr>
</tbody>
</table>

Best-response strategies

- Suppose you know your opponent’s mixed strategy
  - E.g., your opponent plays rock 50% of the time and scissors 50%
- What is the best strategy for you to play?
  - Rock gives .5*0 + .5*1 = .5
  - Paper gives .5*1 + .5*(-1) = 0
  - Scissors gives .5*(-1) + .5*0 = -.5
- So the best response to this opponent strategy is to (always) play rock

There is always some pure strategy that is a best response (given you know the opponent’s strategy)
  - Suppose you have a mixed strategy that is a best response; then every one of the pure strategies that that mixed strategy places positive probability on must also be a best response to some choice.

How to play matching pennies

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<thead>
<tr>
<th></th>
<th>Them</th>
<th>Us</th>
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<tbody>
<tr>
<td>L</td>
<td>1, -1</td>
<td>-2, 2</td>
</tr>
<tr>
<td>R</td>
<td>-1, 1</td>
<td>1, -1</td>
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<table>
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<th>Us</th>
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</tr>
<tr>
<td>R</td>
<td>-2, 2</td>
<td>1, -1</td>
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Matching pennies with a sensitive target

- If we play 50% L, 50% R, opponent will attack L
  - We get .5*(1) + .5*(-1) = .0
- What if we play 55% L, 45% R?
- Opponent has choice between
  - L: gives them .55*(-1) + .45*1 = .35
  - R: gives them .55*(-1) + .45*1 = .1
- We get -.35 > -.5

Support of $\alpha_i$

- All $\alpha_k$ in the mixed strategy for which $p_k > 0$

Mixed strategy NE are necessarily weak. (SLB defn 3.3.6) Why?
Recall a strict NE is when if you deviate from the strategy, you do worse.

How do we compute $p$ (probability we select L)

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<tbody>
<tr>
<td>L</td>
<td>1, −1</td>
<td>1, −1</td>
</tr>
<tr>
<td>R</td>
<td>2, 1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

- Pick $p$ so that our opponent has no preferences between choices
  - $-1(p) + 2(1-p) = 1(p) -1(1-p)$
  - $2-3p = 2p -1$
  - $3 = 5p$
  - $p = .6$
Matching pennies with a sensitive target

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Us</strong></td>
<td>-2, 2</td>
<td>1, -1</td>
</tr>
<tr>
<td><strong>Them</strong></td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

- Our min is -1
- Our min is -2
- What if we play 60% L, 40% R?
- Opponent has choice between
  - L: gives them .6*(-1) + .4*(2) = .2
  - R: gives them .6*(1) + .4*(-1) = .2
- We get -.2 either way
- A mixture of L and R (by our opponent) gives us the same thing as well!
- This is the maximin strategy (we don’t know what the opponent will do, but we assume they will try to minimize our utility. We pick so that we maximize our minimum)

Let’s change roles

<table>
<thead>
<tr>
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<th>L</th>
<th>R</th>
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<tbody>
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<td><strong>Us</strong></td>
<td>-2, 2</td>
<td>1, -1</td>
</tr>
<tr>
<td><strong>Them</strong></td>
<td>1, -1</td>
<td>-1, 1</td>
</tr>
</tbody>
</table>

- Suppose we know their strategy
- If they play 50% L, 50% R,
  - We play L, we get .5*(-1) + .5*(1) = 0
  - If they play 40% L, 60% R,
    - If we play L, we get .4*(-1) + .6*(1) = -2
    - If we play R, we get .4*(2) + .6*(-1) = -2
- This is the minmax strategy (I picks his strategy to minimize the maximum of what the opponent earns)

Can you find?

- Can you find a zero sum game in which maximin (I pick a strategy knowing they will try to minimize what I get) and minmax (I try to minimize the maximum of what they get, and then I’ll get the opposite) causes a different outcome value?

Minimax theorem [von Neumann 1927]

- Maximin utility: $\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i})$
- Minimax utility: $\min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$

- Minimax theorem: in any finite two player, zero sum game, any Nash Equilibrium receives payoff equal to
  $$\max_{\sigma_i} \min_{s_{-i}} u_i(\sigma_i, s_{-i}) = \min_{\sigma_{-i}} \max_{s_i} u_i(s_i, \sigma_{-i})$$

Maximin is i’s best choice when first i must commit to a (possibly mixed) strategy and then the remaining agents observe the strategy (but not the actual choice) and choose to minimize i’s payoff

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<th>a</th>
<th>b</th>
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<tbody>
<tr>
<td><strong>Example</strong></td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Mixed strategy yields equilibrium:
A row player plays (2/3, 1/3), column player (1/3, 2/3)
Then 2/3 is minimax value and deviating doesn’t help.
If I picked A, opponent could get 1.
If I picked B, opponent could get 2.
Practice games
What is the mixed equilibrium?

<table>
<thead>
<tr>
<th></th>
<th>20, -20</th>
<th>0, 0</th>
<th>10, -10</th>
</tr>
</thead>
<tbody>
<tr>
<td>20, -20</td>
<td>0, 0</td>
<td>10, -10</td>
<td>8, -8</td>
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</table>

Solving for minimax strategies using linear programming

- maximize \( u_i \)
- subject to
  - for any \( s_i, \sum s_i p_s u(s_i, s_j) \geq u_i \)
  - \( \sum s_i p_s = 1 \)

Can also convert linear programs to two-player zero-sum games, so they are equivalent

Nash equilibrium [Nash 50]

- A vector of strategies (one for each player) is called a strategy profile
- A strategy profile \((\sigma_1, \sigma_2, \ldots, \sigma_n)\) is a Nash equilibrium if each \( \sigma_i \) is a best response to \( \sigma_{-i} \)
  - That is, for any \( \sigma_i', u(\sigma_i, \sigma_{-i}) \geq u(\sigma_i', \sigma_{-i}) \)
  - Note that this does not say anything about multiple agents changing their strategies at the same time
- In any (finite) game, at least one Nash equilibrium (possibly using mixed strategies) exists [Nash 50]
- (Note - singular: equilibrium, plural: equilibria)

Nash equilibria of “chicken”

- \((D, S)\) and \((S, D)\) are Nash equilibria
  - They are pure-strategy Nash equilibria: nobody randomizes
  - They are also strict Nash equilibria: changing your strategy will make you strictly worse off
- No other pure-strategy Nash equilibria
Nash equilibria of “chicken”...

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>D</td>
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</tr>
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<td>S</td>
<td>1, -1</td>
<td>-5, -5</td>
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</table>

- Is there a Nash equilibrium that uses mixed strategies? Say, where player 1 uses a mixed strategy?
- Recall: if a mixed strategy is a best response, then all of the pure strategies that it randomizes over must also be best responses
- So we need to make player 1 indifferent between D and S
- Player 1’s utility for playing D = -(1-q)
- Player 1’s utility for playing S = q -5(1-q) = 6q -5
- So we need -(1-q) = 6q -5 so q = 4/5
- Then, player 2 needs to be indifferent as well
- Mixed-strategy Nash equilibrium: ((4/5 D, 1/5 S), (4/5 D, 1/5 S))
- People may die! Expected utility = -1/5 for each player

Meaning of Nash Equilibrium

- Does it mean is it the ideal?
- Does it mean it will happen in real life?

Focal Point

- Example: two people unable to communicate with each other are each shown a panel of four squares and asked to select one; if and only if they both select the same one, they will each receive a prize. Three of the squares are blue and one is red. Assuming they each know nothing about the other player, but that they each do want to win the prize, then they will, reasonably, both choose the red square. Of course, the red square is not in a sense a better square; they could win by both choosing any square. And it is the “right” square to select only if a player can be sure that the other player has selected it; but by hypothesis neither can.

Wikipedia Aug 2010

The presentation game

<table>
<thead>
<tr>
<th>Audience</th>
<th>Presenter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pay attention (A)</td>
<td>Do not put effort into presentation (NE)</td>
</tr>
<tr>
<td>4, 4</td>
<td>-16, -14</td>
</tr>
<tr>
<td>0, -2</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

- Pure-strategy Nash equilibria: (A, E), (NA, NE)
- Mixed-strategy Nash equilibrium: ((1/10 A, 9/10 NA), (4/5 E, 1/5 NE))
- Utility 0 for audience, -14/10 for presenter
- Can see that some equilibria are strictly better for both players than other equilibria, i.e. some equilibria Pareto-dominate other equilibria

Focal Point

- In game theory, a focal point (also called Schelling point) is a solution that people will tend to use in the absence of communication, because it seems natural, special or relevant to them. Schelling describes “focal point[s] for each person’s expectation of what the other expects him to expect to be expected to do.” This type of focal point later was named after Schelling.

Tell two people: Pick one of the squares. Win a prize if both of you pick same square.

The “equilibrium selection problem”

- You are about to play a game that you have never played before with a person that you have never met
- According to which equilibrium should you play?
- Possible answers:
  - Equilibrium that maximizes the sum of utilities (social welfare)
  - Or, at least not a Pareto-dominated equilibrium
  - So-called focal equilibria
    - “Meet in Paris” game - you and a friend were supposed to meet in Paris at noon on Sunday, but you forgot to discuss where and you cannot communicate. All you care about is meeting your friend. Where will you go?
  - Equilibrium that is the convergence point of some learning process
  - An equilibrium that is easy to compute
  - …
- Equilibrium selection is a difficult problem
Some properties of Nash equilibria

- If you can eliminate a strategy using strict dominance or even iterated strict dominance, it will not occur (i.e., it will be played with probability 0) in every Nash equilibrium
  - Weakly dominated strategies may still be played in some Nash equilibrium

How hard is it to compute one (any) Nash equilibrium?

- Complexity was open for a long time
  - [Papadimitriou STOC01]: “together with factoring […] the most important concrete open question on the boundary of P today”
- Recent sequence of papers shows that computing one (any) Nash equilibrium is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in the worst case).

What is tough about finding a mixed NE?

- It is not easy to tell which strategies appear in the mix. If you knew the support, the computation is easy.
- The difficulty is in figuring out which strategies are part of the mixed strategy.
- If there are n strategies, 2^n possible supports.
- If one strategy is dominated, we can’t make the player not care between this strategy and others.

Search-based approaches (for 2 players)

- Suppose we know the support \( X_i \) of each player i’s mixed strategy in equilibrium
  - Support: the pure strategies which receive positive probability
- Then, we have a linear feasibility problem:
  - for both i, for any \( s_i \in S_i \setminus X_i \), \( p_i(s_i) = 0 \)
  - for both i, for any \( s_i \in X_i \), \( \sum_{s_j \in S_j \setminus X_j} p_j(s_j) u_i(s_i, s_j) = u_i \)

Search-based approaches (for 2 players)

- Thus, we can search over possible supports
  - This is the basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GBE08]
- Dominated strategies can be eliminated

What if we want to compute a Nash equilibrium with a specific property?

- For example:
  - An equilibrium that is not Pareto-dominated
  - An equilibrium that maximizes the expected social welfare (= the sum of the agents' utilities)
  - An equilibrium that maximizes the expected utility of a given player
  - An equilibrium that maximizes the expected utility of the worst-off player
  - An equilibrium in which a given pure strategy is played with positive probability
  - An equilibrium in which a given pure strategy is played with zero probability
  - …
- All of these are NP-hard (and the optimization questions are inapproximable assuming \( P \neq NP \)), even in 2-player games [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]
Solving for a Nash equilibrium using MIP (2 players)
[Sandholm, Gilpin, Conitzer AAAI05]

- maximize \textit{whatever you like} (e.g., social welfare)
- subject to
  - for both \textit{i}, for any \( s_i \), \( \sum_{s_{-i}} p_{s_{-i}} u_{i}(s_i, s_{-i}) = u_{i}(s) \) others choices make me indifferent to my strategies.
  - for both \textit{i}, for any \( s_i \), \( u_{i} \geq u_{i} \) utility of mixed is no worse than any pure strategy utility.
  - for both \textit{i}, for any \( s_i \), \( p_{s_i} \leq b_{s_i} \) I don’t have positive support unless \( b \) says so.
  - for both \textit{i}, for any \( s_i \), \( u_{i} - u_{i} \leq M(1 - b_{s_i}) \) of supported strategies, mixed will be equal to pure strategy
  - for both \textit{i}, \( \sum_{s_i} p_{s_i} = 1 \) legal support values

- \( b_{s_i} \) is a binary variable indicating whether \( s_i \) is in the support, \( M \) is a large number

Correlated equilibrium \cite{Aumann74}
- Suppose there is a trustworthy mediator who has offered to help out the players in the game
- The mediator chooses a profile of pure strategy outcomes, perhaps randomly, then tells each player what her strategy is in the profile (but not what the other players’ strategies are)
- A correlated equilibrium is a distribution over pure-strategy profiles so that every player wants to follow the recommendation of the mediator (if she assumes that the others do as well). It is no better to deviate.
- Every Nash equilibrium is also a correlated equilibrium
  - Corresponds to mediator choosing players’ recommendations independently
- … but not vice versa
- (Note: there are more general definitions of correlated equilibrium, but it can be shown that they do not allow you to do anything more than this definition.)

Terminology
- When there is nothing to maximize, a LP problem becomes a \textit{feasibility} problem rather than an optimization problem.

A correlated equilibrium for “chicken”

<table>
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<tbody>
<tr>
<td>\textbf{D}</td>
<td>(0, 0)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>\textbf{S}</td>
<td>(1, 0)</td>
<td>(-5, -5)</td>
</tr>
</tbody>
</table>

- Why is this a correlated equilibrium? No reason for either to deviate.
- Suppose the mediator tells the row player to Dodge
- From Row’s perspective, the conditional probability that Column was told to Dodge is 20% / (20% + 40%) = 1/3
- So the expected utility of Row player of Dodging is (2/3)*(-1) = -2/3
- But the expected utility of Straight is (1/3)*1 + (2/3)*(-5) = -3
- So Row wants to follow the recommendation
- If Row is told to go Straight, he knows that Column was told to Dodge, so again Row wants to follow the recommendation
- Similar for Column (as is symmetric)

Correlated
- So, for rock paper scissors – how would you correlate?