Homework 5

You may discuss each problem with one or two other persons. Each person must do her or his own write up. Moreover, you may not simply write down a solution and give it to the other person. You may present things to each other on (say) a whiteboard, but the other person should not simply be copying things over. You should acknowledge everyone you worked with, as well as all other sources, on the write up. Assignments are always due immediately at the beginning of class.

Show all your work, but circle your final answers.

1. (Properties of voting rules.) Alice likes to analyze the outcomes of elections; specifically, she is interested in the different outcomes that different voting rules produce on the same votes. To do so, she executes many different rules on the same set of votes, a painstaking process. She likes knowing about properties of voting rules that ease her task. For example, she likes to know which voting rules satisfy the Condorcet criterion, so that if there is a Condorcet winner, she immediately knows that that will be the winner for those rules, without having to go through the trouble of executing each rule individually.

Recently, Alice has become interested in the phenomenon of votes “cancelling out.” Let us say that a set S of votes cancels out with respect to voting rule r if for every set T of votes, the winner (or set of winners if there are ties) that r produces for T is the same as the winner that r produces for S U T. For example, the set of votes \{a > b > c, b > a > c, c > a > b\} cancels out with respect to the plurality rule: each candidate is ranked first once in this set of votes, so it has no net effect on the outcome of the election.

The same set does not cancel out with respect to Borda, though, because from these votes, a gets 4 points, b gets 3, and c gets 2, which may affect the outcome of the election.

Alice likes to know when a set of votes cancels out with respect to a rule, so that she can just ignore these votes, easing her computation of the winner. Define a pair of opposite votes to be a pair of votes with completely opposite rankings of the candidates, i.e. the votes can be written as c1 > c2 > . . . > cm and cm > cm−1 > . . . > c1. Let us say that a voting rule r satisfies the Opposites Cancel Out (OCO) criterion if every pair of opposite votes cancels out with respect to r.

1a. (12 points) From among the (reasonable) voting rules discussed in class, give 3 voting rules that satisfy the OCO criterion, and 3 that do not (and say which ones are which!). For those that do not cancel out, show a case which illustrates that cancelling out changes the answer.

We define a reasonable voting rule to be (i) not dictatorial rules, (ii) not rules for which there is a candidate that can’t possibly win and (iii) not randomized rules. Also, approval voting cannot be one of the rules because it is not based on rankings. If you use Cup, Cup only satisfies a criterion if it satisfies it for every way of pairing the candidates.
Define a cycle of votes to be a set of votes that can be written as \( c_1 > c_2 > \ldots > c_m, c_2 > c_3 > \ldots c_m > c_1 > c_4 > \ldots > c_m > c_1 > c_2 > \ldots > c_m - 1 \). Let us say that a voting rule \( r \) satisfies the Cycles Cancel Out (CCO) criterion if every cycle cancels out with respect to \( r \).

1b. From among the (reasonable) voting rules discussed in class, give 3 voting rules that satisfy the CCO criterion, and 3 that do not. For those that do not cancel out, show a case which illustrates that cancelling out changes the answer.

Define a pair of opposite cycles of votes to be a cycle, plus all the opposite votes of votes in that cycle (note that these opposite votes themselves constitute a cycle). Let us say that a voting rule \( r \) satisfies the Opposite Cycles Cancel Out (OCCO) criterion if every pair of opposite cycles cancels out with respect to \( r \).

1c. From among the (reasonable) voting rules discussed in class, give 5 voting rules that satisfy the OCCO criterion, and 1 that does not. For those that do not cancel out, show a case which illustrates that cancelling out changes the answer.

2. Not Survivor\(^\text{™} \). Six CS 7910 students Ann, Bob, Carl, Dora, Ed, and Fran are the contestants in a new "reality TV" show. They are placed on an island. The rules of the game are as follows. Ann goes first. She is given a bag that everyone knows contains six gold coins. Ann makes a proposal of how to allocate the six coins among the six contestants including herself. The contestants (including Ann) then vote `yes' or `no' on the proposal. If the proposal gets more than half the votes then the coins are allocated according to the proposal and everyone leaves the island. If the proposal gets half or fewer than half the votes then Ann has to leave the island empty-handed and she is out of the game.

In this case, the bag of six gold coins passes to Bob. He gets to make a proposal of how to allocate the coins among the remaining contestants (i.e., including Bob but excluding Ann) and the remaining contestants (i.e., including Bob but excluding Ann) then vote. As before, if the proposal gets more than half the votes then the coins are allocated according to the proposal and everyone leaves the island. If the proposal gets half or fewer than half the votes then Bob has to leave the island empty-handed and is out of the game. In this case, the bag of six gold coins passes to Carl. And so on, with the same voting rules, with each failed proposal leading to expulsion of the proposer, and with the role of proposer being passed on alphabetically. The following assumptions matter. The coins are indivisible, there is no other money on the island, and side contracts to make payments off the island are not allowed. There are no abstentions; each surviving voter must vote yes or no: whenever a voter is indifferent, she or he votes no. The players only care about the gold (and this is common knowledge). For example, leaving empty handed because your proposal fails is the same as leaving empty-handed because a successful proposal gives you no coins. What proposal should Ann make and why?