1. Show the for each agent \(i\) (with valuation \(v_i\)) in a second-price sealed bid auction with imperfect information, the bid \(v_i\) weakly dominates all other bids.

2. An auctioneer is selling five items \(M = \{a, b, c, d, e\}\). There are seven bids,
\[
A_1 = \{(a, b), 9\}, \\
A_2 = \{(b, e), 12\}, \\
A_3 = \{(c, d, e), 10\}, \\
A_4 = \{(c), 6\}, \\
A_5 = \{(a, c), 13\}, \\
A_6 = \{(b, c, e), 16\}, \\
A_7 = \{(b, d, e), 18\}.
\]

2a. Draw a graphical representation (of your choice) of the bids. Explain your representation.

2b. The auctioneer wants to find the “Optimal Winners”. Who should win?

2c. What do they pay using Vickrey Clark Groves?

3. A multi-unit auction with externalities.
We are running a multi-unit auction for badminton rackets in the town Externa, where nobody owns one yet and we are the only supplier. Of course, being the only person to own a badminton racket is no fun; bidders care about which other bidders win rackets as well. In such a setting, where bidders care about what other bidders win, we say that there are externalities. Let us assume that each agent is awarded at most one racket, and that shuttlecocks and nets are freely available. In the most general bidding language for this setting, each bidder would specify, for every subset of the agents, what her value would be if exactly the agents in that subset won rackets. This is impractical because there are exponentially many subsets. Instead, we will consider more restricted bidding languages.

Let us suppose that it is commonly known which agents live close enough to each other that they could play badminton together. This can be represented as a graph, which has an edge between two agents if and only if they live close enough to each other to play together.

![Externas proximity graph](image)

Figure 1: Externa’s proximity graph.

In the first bidding language, every agent \(i\) submits a single value \(v_i\). The semantics of this are as follows. If the agent does not win a racket, her utility is 0 regardless of who else wins a racket. If she does win a racket, her utility is \(v_i\) times the number of her neighbors that also win a racket. Suppose we receive the following values:
Suppose we have four rackets for sale. One valid (but not optimal) allocation would be to give rackets to Carol, Daniel, and Eva, and Bob. Carol would get a (reported) utility of 6, Daniel would get 8 (2*4, because two of Daniel’s neighbors have rackets), Eva 5, and Bob 4, for a total of 23.

3a. Give the optimal allocation (in terms of maximizing total utility), as well as the VCG payment for each agent. Does VCG make sense in this environment? Recall that to find my payment VCG looks at the utility possible for the system without me minus the utility to others in the winning allocation (that contains me).

3b. In general, is the problem of finding the optimal allocation solvable in polynomial time, or NP-hard? Explain.

One year has passed, and we have returned to Externa. Everyone’s rackets have broken and they need new ones. However, the people in the town were not entirely happy with our previous system. Specifically, it turned out that each agent only ever played with (at most) a single other agent, so that multiplying the value by the number of neighbors with rackets really made no sense. Also, agents have realized that they would receive different utilities for playing with different agents.

In the new system, we must not only decide on who receives rackets, but (for the agents who win rackets) we must also decide on the pairing, i.e., who plays with whom. Each agent can be paired with at most one other agent. Each agent $i$ submits a value $v_{ij}$ for every one of her neighbors $j$; agent $i$ receives $v_{ij}$ if she is paired with $j$ (and both win rackets), and 0 otherwise. Suppose we receive the following bids:

Suppose we have four rackets for sale. One valid (but not optimal) outcome would be to pair Alice and Bob, and Daniel and Eva (and give them all rackets), for a total utility of $3 + 2 + 1 + 2 = 8$. Using VCG (which doesn't make sense for a non-optimal allocation), D would be expected to pay 7 (best possibility without D) minus 7 (utility to others with D – since D only gets 1, others get the rest) = 0.

3c. Give the optimal outcome (pairing and allocation), as well as the VCG (Clarke) payment for each agent. Recall that to find my payment VCG looks at the utility possible for the system without me minus the utility to others in the winning allocation (that contains me).

3d. Is this problem easier or harder than the previous problem?