An Inequality on the Edge Lengths of Triangular Meshes

Minghui Jiang
Utah State University

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For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum length of an edge in $A$ is at least the minimum distance between two vertices in $B$. 

$A \quad B$
Quality Measures for Triangular Meshes

Most previous algorithms for mesh generation either maximize the minimum angle or minimize the maximum angle of a triangular mesh.

The triangular meshes generated by such algorithms bounded maximum-to-minimum angle ratios, but not bounded maximum-to-minimum edge length ratios.
Length-uniform Triangular Meshes


Given a convex polygon $P$ and a positive integer $n$, triangulate $P$ using $n$ Steiner points:

1. minimize the maximum-to-minimum edge length ratio,
2. minimize the maximum edge length,
3. minimize the maximum triangle perimeter.
Aurenhammer et al. proposed an algorithm that, with some reasonable assumptions on the input, achieves a constant approximation ratio for each of the three criteria.

1. first apply a dispersion heuristic to select the Steiner points,

2. next construct the Delaunay triangulation of the polygon using the selected Steiner points,

3. then modify the Steiner triangulation into a triangular mesh.
The Analysis

For a convex polygon $P$ and a positive integer $n$,

$$d_{\text{long}} = \min_T \max_{e \in E(T)} \text{length}(e),$$

where $T$ ranges over all Steiner triangulations of $P$ with $n$ Steiner points, and $E(T)$ is the set of edges in the triangulation $T$.

$$d^* = \max_S \min_{u,v \in S \cup V(P)} \text{distance}(u,v),$$

where $S$ ranges over all sets of $n$ Steiner points in $P$, and $V(P)$ is the set of vertices of the polygon $P$.

The approximation ratios of Aurenhammer et al.’s algorithm crucially depend on the ratio of the two numbers.
The Conjecture

By a simple area argument, Aurenhammer et al. proved that

\[ d_{\text{long}} \geq \frac{\sqrt{3}}{2} d^*, \]

and conjectured that

\[ d_{\text{long}} \geq d*. \]
Conjecture Confirmed by Theorem

For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum length of an edge in $A$ is at least the minimum distance between two vertices in $B$.

$$d_{\text{long}} = \min_T \max_{e \in E(T)} \text{length}(e),$$

$$d^* = \max_S \min_{u,v \in S \cup V(P)} \text{distance}(u,v),$$

$$d_{\text{long}} \geq d^*.$$
$P, S, T \rightarrow A, B, C$

- Let $S$ be a set of $n$ Steiner points in $P$ that realizes the minimum pairwise distance $d^*$ among the point set $S \cup V(P)$.

- Let $T$ be a Steiner triangulation of $P$ with $n$ Steiner points such that the maximum edge length is $d_{\text{long}}$.

$S$ and $T$ are guaranteed by the definitions of $d^*$ and $d_{\text{long}}$.

- Let $C$ be the convex polygon $P$.

- Let $A$ be $T$.

- Let $B$ be any triangulation of the point set $S \cup V(P)$ such that the minimum edge length is $d^*$.
Initialize $B$ to a triangle of any three points in $S$ including a closest pair, then add the remaining points one by one:

- If the point is inside some triangle, add three new edges to split the triangle into three triangles.

- If the point is on some edge, split the edge into two edges, then add additional edges if necessary to split the adjacent triangles.

- If the point is outside all triangles, add a new edge from the point to each visible vertex.

Note that the initial edge between the closest pair is never split and remains in the mesh.
Simplex and Simplicial Complex

A $d$-simplex is the convex hull of $d + 1$ affinely independent vertices (that is, $d + 1$ points in general position) in some Euclidean space of dimension $d$ or higher.

A simplicial complex is a set $K$ of simplices such that

1. any face of a simplex in $K$ is also a simplex in $K$,

2. the intersection of any two simplices $\sigma$ and $\tau$ in $K$ is a face of both $\sigma$ and $\tau$.

A simplex $\sigma$ is a face of another simplex $\tau$ if the vertices of $\sigma$ are a subset of the vertices of $\tau$. 
For a simplicial complex $K$ in the plane, where a 0-simplex is a point, a 1-simplex is a line segment, and a 2-simplex is a triangle, denote by $\alpha_r(K)$ the number of $r$-simplices in $K$, $0 \leq r \leq 2$. Define the Euler characteristic of $K$ as $\chi(K) = \alpha_0(K) - \alpha_1(K) + \alpha_2(K)$.

$$\begin{align*}
\alpha_0(A) &= 4, \quad \alpha_1(A) = 6, \quad \alpha_2(A) = 3, \\
\alpha_0(B) &= 4, \quad \alpha_1(B) = 5, \\
\alpha_2(B) &= 2, \text{ and } \chi(A) = \chi(B) = 1.
\end{align*}$$
Area and Perimeter

For a 1-simplex $\sigma$, denote by $|\sigma|$ the length of $\sigma$, and denote by $\varepsilon(\sigma, K)$ the number of 2-simplices in $K$ having $\sigma$ as a face. Then $\varepsilon(\sigma, K) = 0, 1, \text{ or } 2$.

Define the area of $K$ as the total area of the 2-simplices in $K$. Define the perimeter of $K$ as $\sum_\sigma (2 - \varepsilon(\sigma, K)) |\sigma|$, where $\sigma$ ranges over all 1-simplices in $K$. 
Triangular Mesh As Simplicial Complex

- 2-simplices are triangles,
- 1-simplices are edges,
- 0-simplices are polygon vertices and Steiner points.
- The Euler characteristic of the triangular mesh is exactly one minus the number of holes in the polygon.
- The area and the perimeter of the triangular mesh (as a simplicial complex) are respectively the same as the area and the perimeter (in the normal sense) of the polygon.
The Proof

For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum length of an edge in $A$ is at least the minimum distance between two vertices in $B$.

Denote by $\ell_{\text{max}}(T)$ and $\delta_{\text{min}}(T)$, respectively, the maximum length of an edge and the minimum distance between two vertices in a triangular mesh $T$. We will prove the contrapositive:

Let $A$ and $B$ be two triangular meshes of the same polygon $C$. Suppose that $\delta_{\text{min}}(B) > \ell_{\text{max}}(A)$. Then $\alpha_0(B) < \alpha_0(A)$. 
The 40-Year Old Lemma

Our proof will use the following lemma by Folkman and Graham [Canadian Mathematical Bulletin (1969), 745–752]:

Let $K$ be a simplicial complex in the plane. Suppose that the distance between any two 0-simplices in $K$ is at least 1. Then the total number of 0-simplices in $K$ is at most $\frac{2}{\sqrt{3}} \text{area}(K) + \frac{1}{2} \text{peri}(K) + \chi(K)$.

In other words, if $\delta_{\text{min}}(K) \geq ?$, then $\alpha_0(K) \leq ???$.

Our goal: if $\delta_{\text{min}}(B) > \ell_{\text{max}}(A)$, then $\alpha_0(B) < \alpha_0(A)$.

To use this lemma, we could let $B = K$, then try to relate $\alpha_0(A)$ to the area, perimeter, and Euler characteristic of $B$ (or of $A$).
Bounding the Area of $A$

For any triangle of edge lengths at most $\ell$, we can transform it into an equilateral triangle of edge length exactly $\ell$ as follows.

1. First move any vertex of the triangle perpendicularly away from the opposite edge, until one of the two edges incident to the vertex has length exactly $\ell$,

2. next extend the other incident edge until its length is also $\ell$,

3. and finally extend the opposite edge also to length $\ell$.

Note that the area of the triangle does not decrease during this transformation. Since an equilateral triangle of edge length $\ell$ has an area exactly $\frac{\sqrt{3}}{4} \ell^2$, the area of $A$ is at most $\alpha_2(A) \cdot \frac{\sqrt{3}}{4} \ell_{\max}^2(A)$. 
Bounding the Perimeter of $A$

Denote by $\beta_0(T)$ the number of vertices in a triangular mesh $T$ that are on the boundary of the underlying polygon, including the polygon vertices and possibly additional Steiner vertices on the boundary.

Since $A$ has exactly $\beta_0(A)$ edges on the boundary of $C$, the perimeter of $A$ is at most $\beta_0(A) \cdot \ell_{\text{max}}(A)$.

\[ \beta_0(A) = 3 \text{ and } \beta_0(B) = 4. \]
\[ \alpha_2(A), \alpha_0(A), \beta_0(A), \text{ and } \chi(A) \]

Each boundary edge of a triangular mesh is incident to one triangle. Each internal edge of a triangular mesh is incident to two triangles. Each triangle has three edges. Thus by double-counting we have

\[
1 \cdot \beta_0(A) + 2 \cdot (\alpha_1(A) - \beta_0(A)) = 3 \cdot \alpha_2(A)
\]

\[
\implies \alpha_1(A) = \frac{3\alpha_2(A) + \beta_0(A)}{2}.
\]

Recall that \( \chi(A) = \alpha_0(A) - \alpha_1(A) + \alpha_2(A) \). Thus

\[
\chi(A) = \alpha_0(A) - \frac{3\alpha_2(A) + \beta_0(A)}{2} + \alpha_2(A)
\]

\[
\implies \alpha_2(A) = 2\alpha_0(A) - \beta_0(A) - 2\chi(A).
\]
\[ \alpha_0(B) \]
\[ \leq \frac{2}{\sqrt{3}} \cdot \frac{\text{area}(B)}{\delta_{\min}^2(B)} + \frac{1}{2} \cdot \frac{\text{peri}(B)}{\delta_{\min}(B)} + \chi(B) \quad \text{(by Lemma)} \]
\[ = \frac{2}{\sqrt{3}} \cdot \frac{\text{area}(A)}{\delta_{\min}^2(B)} + \frac{1}{2} \cdot \frac{\text{peri}(A)}{\delta_{\min}(B)} + \chi(A) \]
\[ < \frac{2}{\sqrt{3}} \cdot \frac{\text{area}(A)}{\ell_{\max}^2(A)} + \frac{1}{2} \cdot \frac{\text{peri}(A)}{\ell_{\max}(A)} + \chi(A) \]
\[ \leq \frac{2}{\sqrt{3}} \cdot \frac{\alpha_2(A) \cdot \frac{\sqrt{3}}{4} \ell_{\max}^2(A)}{\ell_{\max}^2(A)} + \frac{1}{2} \cdot \frac{\beta_0(A) \cdot \ell_{\max}(A)}{\ell_{\max}(A)} + \chi(A) \]
\[ = \frac{1}{2} \alpha_2(A) + \frac{1}{2} \beta_0(A) + \chi(A) \]
\[ = \frac{1}{2} \left( 2\alpha_0(A) - \beta_0(A) - 2\chi(A) \right) + \frac{1}{2} \beta_0(A) + \chi(A) = \alpha_0(A). \]
A Variation?

We have proved the following:

For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum length of an edge in $A$ is at least the minimum distance between two vertices in $B$.

A natural question is whether the following is also true:

For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum edge length of $A$ is at least the minimum edge length of $B$. 
\[ \ell_{\text{max}}(A) \geq \ell_{\text{min}}(B) = \delta_{\text{min}}(B) \]
A Counter-example

Two triangular meshes $A$ and $B$ of the same unit equilateral triangle $C$ such that $A$ and $B$ have the same number of vertices but every edge of $B$ is longer than every edge of $A$. 
From $A$ to $A'$

$A$ has $(5+1)(5+2)/2 = 21$ vertices and uniform edge length $1/5$. $A'$ has $(4+1)(4+2)/2 = 15$ vertices and uniform edge length $1/4$. 
From $A'$ to $A''$ and $Z$
From $A''$ and $Z$ to $B$

$B$ has $15 + 2 \cdot 3 = 21$ vertices and minimum edge length $\approx \frac{1}{4}$. 
A and B

A has 21 vertices and uniform edge length $1/5$.

B has 21 vertices and minimum edge length $\approx 1/4$. 

The Construction Can Be Generalized

For each $k \geq 5$, there are two triangular meshes $A$ and $B$ of the unit equalateral triangle such that

- $A$ has $(k + 1)(k + 2)/2$ vertices and uniform edge length $1/k$.
- $B$ has $k(k + 1)/2 + 3(k - 3) = (k + 1)(k + 2)/2 + 2(k - 5)$ vertices and minimum edge length close to $1/(k - 1)$. 
Another False Statement

The following statement is false too:

For any two triangular meshes $A$ and $B$ of the same polygon $C$, if the number of vertices in $A$ is at most the number of vertices in $B$, then the maximum triangle perimeter of $A$ is at least the minimum triangle perimeter of $B$. 
A Counter-example

Each triangular mesh has four vertices: three vertices of the triangle and one Steiner point.

Maximum triangle perimeter of $A$: $1 + 2\sqrt{\frac{3}{3}} = 2.15 \ldots$

Minimum triangle perimeter of $B$: $1 + \frac{1}{2} + \frac{\sqrt{3}}{2} = 2.36 \ldots$
An updated version of the paper is online at

http://www.cs.usu.edu/~mjiang/

Thank you!