CONNECT-THE-DOTS IN A GRAPH AND BUFFON'S NEEDLE ON A CHESSBOARD: TWO PROBLEMS IN ASSISTED NAVIGATION

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We study two theoretical problems that arise naturally in the application domain of assisted navigation. Connect-the-dots in a graph is a graph-theoretical problem with application to robot indoor localization. Buffon's needle on a chessboard is a problem in geometric probability with application to the design of RFID-enabled surface for robot-assisted navigation.

Keywords: Graph algorithms; Geometric probability; Assisted navigation.

1. Connect-the-Dots in a Graph

Aging adults have increasing difficulties in everyday activities due to the decline in their sensory-motor abilities, and need smart robotic devices for assisted navigation.[4,6] An important aspect of a navigation device is its *localization* ability, that is, the ability to determine its own location when carried around by a user. Although numerous localization methods are available, such as Markov localization,[2] RFID-based localization,[5] and wireless localization,[7] each method has its own vulnerability. We take the wireless localization method[7] as an example. As the wireless network technology becomes ubiquitous, an indoor environment often contains many wireless access points. The aggregate wireless signals from these access points have different patterns at different locations. The wireless localization method[7] uses machine learning algorithms to classify the locations based on the received wireless signals; it can make mistakes when the environment is noisy.

A user equipped with a navigation device often has a limited speed of movement. It is therefore reasonable to assume that the current location of the device is always "close" to its most recent location. This suggests a possible way to improve the accuracy of a localization method by correcting certain mistakes that are apparent from the location history and the environment geometry.

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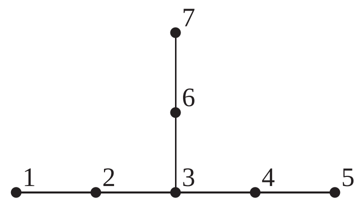

Fig. 1. Connect the dots in a graph.

We model the indoor environment as a graph: the vertices correspond to the discrete locations in the environment; the edges correspond to the paths, the stairs, and the corridors that connect the discrete locations. In this graph, the assumption that the current location of the device is "close" to a recent location can be formulated as a constraint: two consecutive locations must correspond to either the same vertex or two adjacent vertices in the graph. We refer to Fig. 1 for an example. If a localization method reports the sequence of locations $\langle 1, 2, 7, 4, 5\rangle$, then 7 is probably misclassified and may be corrected to 3.

We now introduce a graph-theoretical problem motivated by our idea for improving the accuracy of localization methods. For a sequence S, denote by $|S|$ the length of S, denote by $S[i]$ the i'th character of S, and denote by $S[i, j]$ as the subsequence $S[i] \ldots S[j]$. Define the *distance* between two sequences S and T of the same length l as

$$dist(S, T) = \sum_{i=1}^{l} d(S[i], T[i]),$$

where $d(u, v)$ is 0 if $u = v$ and is 1 otherwise. A sequence S *conforms* to a graph G if

(1) every character of S is a vertex of G;
(2) every two consecutive characters of S are either the same vertex or two adjacent vertices of G.

Definition 1.1 (Connect-the-dots in a graph). *Given a connected graph $G = (V, E)$ and a sequence S, find a sequence T of the same length as S such that T conforms to G and that the distance $dist(S, T)$ is minimized.*

We design a dynamic programming algorithm for the connect-the-dots problem. For simplicity, we only demonstrate how to compute the minimum distance $dist(S, T)$; the actual sequence T that achieves the minimum distance with S can be reconstructed in the same running time using standard techniques in dynamic programming.

Denote by $D[k][v]$ the minimum distance between the subsequence $S[1,k]$ and a graph-conforming sequence of the same length k ending at the vertex $v \in V$. Our algorithm has the base condition

$$D[1][v] = d(S[1], v)$$

and the recurrence

$$D[k+1][v] = \min \begin{cases} D[k][v] + d(S[k+1], v) \\ \min\limits_{\{u,v\} \in E} \Big\{ D[k][u] + d(S[k+1], v) \Big\}. \end{cases}$$

The minimum distance is $\min_{v \in V} D[|S|][v]$.

We give an analysis of the complexities of our algorithm. For each k, the recurrence takes $O(|V|+|E|)) = O(|E|)$ time; the overall time complexity is therefore $O(|E||S|)$. If we only want to compute the minimum distance $\min_{v \in V} D[|S|][v]$, then $O(|V|)$ space is sufficient to store the D table for at most two consecutive values of k. On the other hand, if we want to reconstruct the graph-conforming sequence T with the minimum distance to the sequence S, some additional space is needed to store a partial graph-conforming sequence ending at each vertex $v \in V$ as the algorithm steps through k from 1 to $|S|$, and the space complexity becomes $O(|V||S|)$. We have the following theorem.

Theorem 1.1. *Given a connected graph $G = (V, E)$ a sequence S, there is an $O(|E||S|)$ time and $O(|V||S|)$ space algorithm that finds a sequence T of the same length as S such that T conforms to G and that $dist(S, T)$ is minimized.*

Our algorithm can be easily adapted to either the *weighted* case in which the distance $d(u, v)$ between two vertices u and v is not the Hamming distance but an arbitrary given weight, or the *on-line* case in which the sequence S is given one character at a time. We are conducting experiments to measure the effectiveness of our algorithm on improving the accuracy of various indoor-localization methods. The experimental results are deferred to the full version of this paper.

2. Buffon's Needle on a Chessboard

The Buffon's needle problem, first posed by the French naturalist Buffon in 1733,[1] is probably the best-known problem in the field of geometric probability.[3,8] We now review this classical problem and its solution.

Definition 2.1 (Buffon's needle on parallel lines). *A needle is dropped at random on the plane marked by equidistant parallel lines. What is the probability that the needle cuts (intersects) at least one line?*

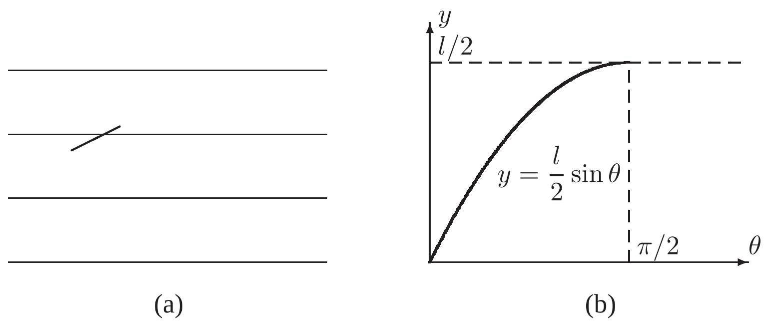

Fig. 2. Buffon's needle on parallel lines. (a) A short needle on parallel lines. (b) The probability.

We refer to Fig. 2. Denote by l the length of the needle, and by h as the distance between two consecutive parallel lines. Let y be the minimum distance from the needle's midpoint to a line, and let θ be the minimum angle formed by the needle and the parallel lines; it is clear that $0 \leq y \leq h/2$ and $0 \leq \theta \leq \pi/2$. Buffon considered the case that $l < h$, such that the needle can intersect at most one line, and interpreted the phrase "at random" to mean that y and θ are independently and uniformly distributed in the ranges $[0, h/2]$ and $[0, \pi/2]$, respectively. Since the needle cuts one of the parallel lines if and only if $0 \leq y \leq (l/2)\sin\theta$, the probability is

$$\Pr\Big(0 \leq y \leq (l/2)\sin\theta\Big) = \frac{\int_0^{\pi/2}\int_0^{(l/2)\sin\theta} \mathrm{d}\theta\, \mathrm{d}y}{(\pi/2)(h/2)} = \frac{2l}{\pi h}.$$

We next consider an extension of the Buffon's needle problem that is motivated by the robot-assisted navigation on RFID-enabled surfaces. An RFID-enabled surface is a surface embedded with a set of passive radio frequency identification sensors. Each RFID sensor has an active area on the surface. When an antenna moves above the active area of a sensor, it receives a uniquely coded signal from the sensor, that is, the antenna gets a *read* of the sensor. We refer to Fig. 3(a). A robot is equipped with two antenna. When an antenna gets a read, the robot can look up the location of the RFID sensor from a table using the unique code in the signal as an index; this location gives a rough estimate of the actual location of the antenna. When both antenna get reads, the robot can compute from the locations of the two antenna its own location and orientation, which are crucial for its navigation on the RFID-enabled surface.

For the ease of implementation, the RFID sensors are often placed in a grid formation. We refer to Fig. 3(b). Because of the electromagnetic interference between the sensors, the aggregate active area, above which the antenna get valid

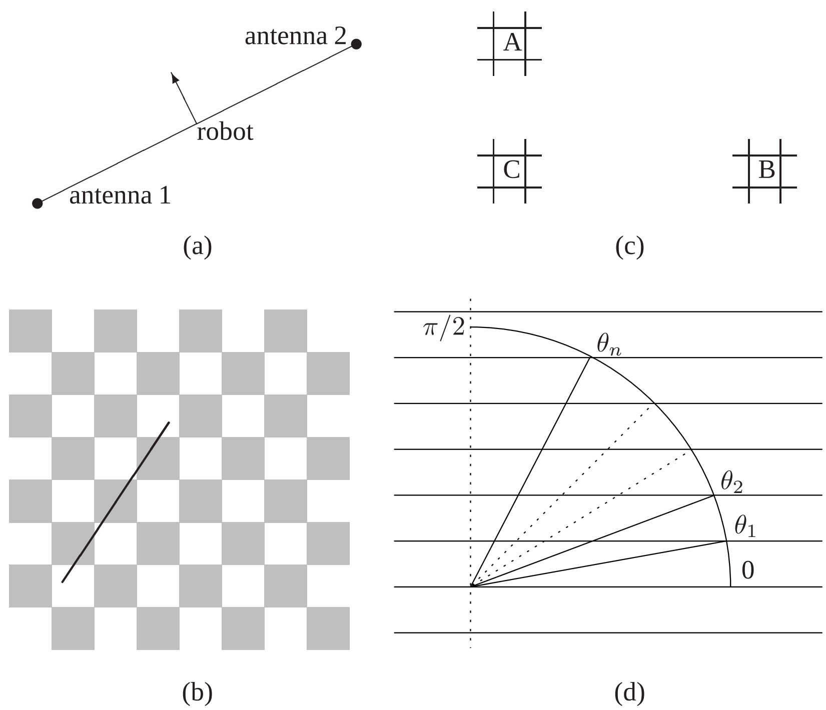

Fig. 3. Buffon's needle on a chessboard. (a) A robot with two antenna. (b) A needle on a chessboard. (c) A and B have the same color if and only if $u + v$ is even. (d) The number of horizontal cuts.

reads, manifests a pattern that resembles the black cells in a chessboard. Since the robot can compute its own location and orientation only when both antenna get reads, that is, when both antenna are in active area, the following problem naturally arises concerning the probability of such an event.

Definition 2.2 (Buffon's needle on a chessboard). *A needle is dropped at random on a chessboard. What the probability that the needle's two endpoints are in two cells of the same color?*

We assume that the chessboard, instead of being eight-by-eight, has an infinite number of cells in alternating black and white colors that tile the entire plane. Clearly, the probability that the needle's two endpoints are both in black cells (the active area) is exactly half the probability that they are in two cells of the same color.

Denote by h the side length of a chessboard cell. Let x and y, $0 \leq x \leq h$ and $0 \leq y \leq h$, be the distances from the needle's midpoint, respectively, to the nearest vertical grid line to the left, and to the nearest horizontal grid line from below. Let θ, $0 \le \theta \le \pi/2$, be the minimum angle formed by the needle and a horizontal line of the chessboard. We interpret the phrase “at random” to mean that x, y, and θ are independently and uniformly distributed in the ranges $[0, h]$, $[0, h]$, and $[0, \pi/2]$, respectively.

Let u and v, respectively, be the numbers of horizontal lines and vertical lines that the needle cuts. Let $n = \lfloor l/h \rfloor$. It is clear that both numbers u and v are in the range $[0, n + 1]$. We have the following proposition.

Proposition 2.1. *The two endpoints of the needle are in two cells of the same color if and only if $u + v$ is even.*

We refer to Fig. 3(c). Let A and B be the two cells containing the two endpoints of the needle. Let C be a cell in the same column as A and in the same row as B. A and C have the same color if and only if the number u of horizontal grid lines between them is even; B and C have the same color if and only if the number v of vertical grid lines between them is even. The correctness of Proposition 2.1 is immediate from the following four cases:

(1) u and v are even: A, B, and C have the same color.
(2) u and v are odd: A and B have the same color; C has a different color.
(3) u is even, v is odd: A and C have the same color; B has a different color.
(4) u is odd, v is even: B and C have the same color; A has a different color.

Therefore, the probability that the two endpoints of the needle are in two cells of the same color is

$$\Pr(\text{same color}) = \sum_{\substack{u,v \in [0, n+1] \\ u+v \text{ is even}}} \Pr(u, v), \tag{1}$$

where $\Pr(u, v)$ denotes the probability that the needle cuts exactly u horizontal lines and v vertical lines.

We now calculate $\Pr(u, v)$ using a standard technique in geometric probability.[8] We refer to Fig. 3(d). Fix one endpoint of the needle on the intersection of a horizontal line and a vertical line, then rotate the needle from $\theta = 0$ to $\theta = \pi/2$. Define $(\theta_{-1}, \theta_0, \ldots, \theta_{n+1}, \theta_{n+2})$ such that $\theta_{-1} = \theta_0 = 0$, $\theta_1 = \arcsin(h/l)$, $\theta_2 = \arcsin(2h/l)$, …, $\theta_n = \arcsin(nh/l)$, and $\theta_{n+1} = \theta_{n+2} = \pi/2$. Since the needle cuts exactly u horizontal lines, the angle θ must be in the range $[\theta_{u-1}, \theta_{u+1})$:

- If $\theta \in [\theta_{u-1}, \theta_u)$, the needle can translate downward for a distance of at most $l \sin\theta - (u - 1)h$ without affecting the number of horizontal cuts.
- If $\theta \in [\theta_u, \theta_{u+1})$, the needle can translate upward for a distance of at most $(u + 1)h - l \sin\theta$.

Symmetrically, since the needle cuts v vertical lines, the angle θ must be in the range $(\pi/2 - \theta_{v+1}, \pi/2 - \theta_{v-1}]$:

- If $\theta \in (\pi/2 - \theta_{v+1}, \pi/2 - \theta_v]$, the needle can translate to the right for a distance at most $(v + 1)h - l\cos\theta$ without affecting the number of vertical cuts.
- If $\theta \in (\pi/2 - \theta_v, \pi/2 - \theta_{v-1}]$, the needle can translate to the left for a distance at most $l\cos\theta - (v - 1)h$.

Define

$$[\alpha_{u,v}, \beta_{u,v}] = [\theta_{u-1}, \theta_{u+1}) \cap (\pi/2 - \theta_{v+1}, \pi/2 - \theta_{v-1}], \tag{2}$$

$$y_u(\theta) = \begin{cases} l\sin\theta - (u-1)h & \text{if } \theta \in [\theta_{u-1}, \theta_u) \\ (u+1)h - l\sin\theta & \text{if } \theta \in [\theta_u, \theta_{u+1}), \end{cases} \tag{3}$$

$$x_v(\theta) = \begin{cases} (v+1)h - l\cos\theta & \text{if } \theta \in (\pi/2 - \theta_{v+1}, \pi/2 - \theta_v] \\ l\cos\theta - (v-1)h & \text{if } \theta \in (\pi/2 - \theta_v, \pi/2 - \theta_{v-1}]. \end{cases} \tag{4}$$

We have

$$\Pr(u, v) = \frac{1}{h^2(\pi/2)} \int_{\alpha_{u,v}}^{\beta_{u,v}} x_v(\theta)\, y_u(\theta)\, \mathrm{d}\theta. \tag{5}$$

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